Int. J. Solids Structures Vol. 23, No. 4, pp. 537–550, 1987
Printed in Great Britain.

0020–7683/87 $3.00+.00
Pergamon Journals Ltd.

LEAST-WEIGHT DESIGN OF PERFORATED ELASTIC PLATES—II

G. I. N. ROZVANY,† T. G. ONG, W. T. SZETO and R. SANDLER
Department of Civil Engineering, Monash University, Clayton, Victoria, Australia 3168

and

N. OLHOFF‡ and M. P. BENDSØE
Department of Solid Mechanics and Mathematical Institute, Technical University of Denmark, DK-2800 Lyngby, Denmark

(Received 21 June 1985)

Abstract—On the basis of a simple specific cost function for perforated, elastic plates with a microstructure of first- and second-order rib systems we derive optimal designs for axially symmetric plates with simply supported, clamped or loaded edges. The results are obtained by variational analysis and the conclusion of the analysis is that for the class of plates considered a first/second-order microstructure does not result in an improved economy as compared to the first-order microstructure.

INTRODUCTION

In Part I of this study, the relative economy of various microstructures was determined, a general formulation for prescribed compliance discussed, a specific cost function derived and the volume of intuitive designs for uniformly loaded perforated circular simply supported plates was compared.

In Part II, a general variational analysis is presented and it is shown that the latter fully confirms the best intuitive design derived in Part I for the case of simply supported plates. The nonuniqueness of the solution for unloaded regions is also investigated and several additional loading/boundary conditions considered.

Part II is in notation and analysis closely related to Part I, and for this reason sections, equations, figures and references are numbered in succession to the numbers used in Part I.

7. VARIATIONAL FORMULATION

We shall perform a variational analysis of the problem of minimizing the weight of compliance-constrained axisymmetric perforated elastic plates described by the micromodel proposed in Section 5 (Part I). Using the notation established in Section 6 of Part I and repeated in the appendix, the problem can be formulated as

$$\min_{S_\theta, S_r, M_\theta, M_r} \Phi = \int_0^1 \psi(S_\theta, S_r) r \, dr \tag{51}$$

subject to

$$\int_0^1 (M_\theta^2/S_\theta + M_r^2/S_r) r \, dr = C, \qquad 0 \leqslant S_i \leqslant 1 \qquad (i = \theta, r)$$

$$(rM_r)'' - M_\theta' = -r, \qquad (M_\theta r/S_\theta)' = (M_r/S_r) \tag{52}$$

and homogeneous boundary conditions, where Φ is the total plate volume, ψ the specific

† Since 1985: FB 10, Essen University, 4300 Essen 1, West Germany.
‡ Since 1985: Department of Engineering Design, Aalborg University, DK-9220 Aalborg, Denmark.

SAS 23:4-G

cost (plate volume per unit area of middle surface), S_θ, S_r stiffnesses in the principal directions, M_θ, M_r the principal moments, r the radial coordinate, C the total compliance and primes denote differentiation with respect to r. Note that we assume constant transverse load over the plate domain (cf. eqn (17)). The last two conditions under eqn (52) represent equilibrium and compatibility. Introducing the Lagrangian functions, $L_1, \ldots, L_4$ and u, the Lagrangian multiplier λ and the slack functions $s_1, \ldots, s_4$, the above problem can be expressed as

$$\min \Phi = \int_D \Big[\psi(S_\theta, S_r) + \lambda(M_\theta^2/S_\theta + M_r^2/S_r) + L_1(-S_\theta + s_1^2) + L_2(-S_r + s_2^2) + L_3(S_\theta - 1 + s_3^2) + L_4(S_r - 1 + s_4^2) + u(M_r'' + 2M_r'/r - M_\theta'/r + 1) \Big] r\,\mathrm{d}r - \lambda C. \tag{53}$$

As will be shown later, compatibility is automatically satisfied by the optimal solution. Then the usual *necessary* conditions of minimality for variations of S_θ, S_r, M_θ, M_r and s_i $(i = 1, \ldots, 4)$ imply

$$\psi_{,S_\theta} = \lambda M_\theta^2/S_\theta^2 + L_1 - L_3 \tag{54}$$

$$\psi_{,S_r} = \lambda M_r^2/S_r^2 + L_2 - L_4 \tag{55}$$

$$-u'/r = 2\lambda M_\theta/S_\theta \tag{56}$$

$$-u'' = 2\lambda M_r/S_r \tag{57}$$

$$L_i \geqslant 0, \qquad s_i L_i = 0 \qquad (i = 1, \ldots, 4). \tag{58}$$

Denoting the elastic plate deflection by $w(r)$, its curvatures by

$$M_\theta/S_\theta = \kappa_\theta = -w'/r, \qquad M_r/S_r = \kappa_r = -w'' \tag{59}$$

and observing that transversality conditions give the same boundary conditions for $w(r)$ and $u(r)$, it can be seen from eqns (56) and (57) that

$$w = u/2\lambda. \tag{60}$$

The deflection w and the Lagrangian multiplier u are proportional since we deal with a self-adjoint problem. Note that by eqns (56), (57) and (59) the elastic compatibility condition $(M_\theta r/S_\theta)' = (M_r/S_r)$ is satisfied. This is to be expected since the stationarity of Φ with respect to the moments is equivalent to the necessary condition that an equilibrium moment field minimizes compliance (cf. the dual variational principle of mechanics). It was not necessary, therefore, to incorporate the compatibility conditions into eqn (53).

Introducing the notation $k = 1/\sqrt{\lambda}$, conditions (54)–(59) and formula (10) for the cost ψ imply that:

for $0 < S_i < 1, \quad 0 < S_j < 1 \quad (i = \theta, r; j = r, \theta)$

$$|\kappa_i| = k\sqrt{\psi_{,S_i}} = k(1 - S_j)/(1 - S_i S_j); \tag{61}$$

for $S_i = 0$, $0 < S_j < 1$

$$M_i = 0, \quad |\kappa_i| \leqslant k(1 - S_j), \quad |\kappa_j| = k, \quad S_j = |M_j|/k; \tag{62}$$

for $S_\theta = S_r = 1$

$$|\kappa_i| = |M_i| \geqslant k\sqrt{\psi_{,S_i}} = 0. \tag{63}$$

As the gradient of $\psi(S_1, S_2)$ is nonunique at $S_1 = S_2 = 1$, its value in the directions of the S_1 and S_2 axes (cf. Fig. 5) was taken $(\partial\psi/\partial S_1 = \partial\psi/\partial S_2 = 0)$.

8. ELIMINATION OF CERTAIN TYPES OF REGIONS FROM THE OPTIMAL SOLUTIONS

8.1. $M_r \equiv S_r \equiv 0$, $S_\theta \neq 0$ *is non-optimal over a finite interval*

In this case eqns (51) and (62) imply that

$$\kappa_\theta = k \operatorname{sgn} M_\theta = -w'/r, \qquad -w' = rk \operatorname{sgn} M_\theta, \qquad -w'' = k \operatorname{sgn} M_\theta = \kappa_r$$
$$\kappa_r \leqslant k(1 - S_\theta) \tag{64}$$

which cannot be satisfied with $S_\theta \neq 0$.

8.2. $M_\theta \equiv S_\theta \equiv 0$ *for* $0 \leqslant r \leqslant a$, $M_r(a) = \bar{M} \geqslant 0$ *cannot be optimal if the load* p *is positive*

From equilibrium we have that

$$rM_r\big|_{r=0} = a\bar{M} + \int_0^a \int_0^a p(r) r \, dr \, dr > 0 \tag{65}$$

and from the necessary conditions (62)

$$S_r = |M_r|/k \qquad \text{for} \quad (0 \leqslant r \leqslant a). \tag{66}$$

However, eqn (65) implies $M_r(0) \to \infty$ and since k (and thus C) cannot be infinite eqn (66) implies that the condition $S_r \leqslant 1$ would not hold.

8.3. $0 < S_\theta < 1$ *and* $0 < S_r < 1$ *is not optimal in a finite interval if* $p(r) \neq 0$

For this case we can rewrite eqns (59) and (61) to obtain

$$\begin{aligned} M_\theta &= k[1 - (1 - S_\theta)/(1 - S_\theta S_r)] \operatorname{sgn} M_\theta \\ M_r &= k[1 - (1 - S_r)/(1 - S_\theta S_r)] \operatorname{sgn} M_r \end{aligned} \tag{67}$$

and

$$\begin{aligned} (rM_r)'' \operatorname{sgn} M_r &= -k[r(1 - S_r)/(1 - S_\theta S_r)]'' \\ &= -(r\kappa_\theta)'' \operatorname{sgn} M_\theta = w''' \operatorname{sgn} M_\theta \\ (M_\theta)' \operatorname{sgn} M_\theta &= -k[(1 - S_\theta)/(1 - S_\theta S_r)]' \\ &= -(\kappa_r)' \operatorname{sgn} M_r = w''' \operatorname{sgn} M_r. \end{aligned} \tag{68}$$

From this we see that

$$(rM_r)'' - M_\theta' = 0 \tag{69}$$

which violates equilibrium for $p(r) \neq 0$.

8.4. $S_i = 1$, $0 < S_j < 1$ *for a finite interval is not optimal*

For these stiffnesses we have $\partial\psi/\partial S_j = 0$, and eqns (54) and (55) give that $\kappa_j = 0$. As $S_i = 1$ we thus have that either $w' = 0$ or $w' = \text{const.}$ over the interval considered, but these deflections are not kinematically admissible.

Note: The analyses above imply that the solution to the optimal design problem may consist only of:

(a) unperforated regions, subsequently termed *R_0-regions*, where $S_r = S_\theta = 1$,
(b) regions with radial ribs only, called *R_r-regions*, where $M_\theta = S_\theta = 0$, $S_r = M_r/k$.

The innermost region ($0 \leqslant r \leqslant g$) must always be an unperforated R_0-region.

9. CONDITIONS FOR OPTIMALLY PLACED R_r-BOUNDARIES AND RADII OF ZERO MOMENTS

9.1. *R_r-boundaries*

Considering boundary g, $0 < g < 1$, between an R_0-region and an R_r-region we seek a condition for the optimal location of g. To this end, we consider eqn (53) for a family of solutions that satisfies the necessary conditions (54)–(58) and constraints (52), but where g has yet to be optimally placed. Thus Φ is a function of g and we can write (only the first two terms in eqn (53) remain)

$$\Phi = \int_s^g \left(1 + \frac{1}{k^2}(\kappa_\theta^2 + M_r^2)\right) r\,\mathrm{d}r + \int_g^t \left(S_r + \frac{1}{k^2}(k|M_r|)\right) r\,\mathrm{d}r + Q - \frac{1}{k^2}C \tag{70}$$

where the R_0-region is $s \leqslant r \leqslant g$ and the R_r-region is $g \leqslant r \leqslant t$, and where Q contains the parts of Φ not relevant to g. Taking the derivative of Φ with respect to g and using that the curvature κ_θ and the moment M_r are continuous across $r = g$ we have

$$\frac{\partial\Phi}{\partial g} = \frac{1}{k^2}[k^2 + \kappa_\theta^2 + M_r^2 - 2k|M_r|] = \frac{1}{k^2}[\kappa_\theta^2 + (k - |M_r|)^2]. \tag{71}$$

We note here, that in R_r-regions we have (cf. eqn (62))

$$|\kappa_\theta| \leqslant k - |M_r| \tag{72}$$

which is thus also a constraint on g. From eqn (71) we have that $\partial\phi/\partial g$ is positive, so the necessary condition for optimality of ϕ with respect to g under constraint (72) requires that this constraint is active

$$|\kappa_\theta| = k - |M_r| \quad \text{at} \quad r = g. \tag{73}$$

Thus, a condition for an optimally placed boundary is that constraint (72) is satisfied as an equality at the boundary.

9.2. *Radii of zero moments*

We now consider a radius $r = h$ for which the moments M_r and M_θ are zero, and seek a condition for the optimal position of such a radius. As in the preceding section we consider the augmented functional Φ for plates that satisfy conditions (54)–(58) and (52). By use of eqns (59) and an integration by parts we can write Φ as

$$\Phi = \int_0^1 \left[\psi + \frac{1}{k^2}\left(-M_\theta \frac{w'}{r} + \left(M_r' + \frac{M_r}{r}\right) w'\right)\right] r\,\mathrm{d}r - \frac{1}{k^2}C. \tag{74}$$

At the radius $r = h$ we have that $M_\theta(h) = M_r(h) = 0$, and for R_0-regions the cost $\psi = 1$ throughout, while for R_r-regions the condition $|\kappa_r| = k$ implies that the cost ψ at h is $S_r(h) = 0$. Writing integral (74) over the intervals $0 \leqslant r < h$ and $h < r \leqslant 1$, it is easily seen that stationarity of Φ with respect to h requires

$$\langle M_r' w' \rangle_h = 0. \tag{75}$$

From the equilibrium condition (18a) we see that M_r' is continuous and different from zero at $r = h$, so eqn (75) implies that

$$w'(h^+) = w(h^-) \tag{76}$$

i.e. w' is continuous at optimally placed radii of zero moments.

10. SOLUTION FOR SIMPLY SUPPORTED PLATES

From the results in Section 8 we know that the optimal solution has the form:

(a) R_0-*region* for $r \leqslant g$, $v \leqslant r \leqslant t$ $(S_r = S_\theta = 1)$,
(b) R_r-*region* for $g < r < v$ $(S_\theta = 0)$,

and perhaps more R_0, R_r-regions for $r \geqslant t$. We shall now show that $v = t = 1$ for the simply supported plates, so such optimal plates contain only one of each of the possible regions.

For the regions above eqns (18a) and (18b) give that (cf. eqns (35)–(39))

$$\begin{aligned} M_r &= A - 3r^2/16 \quad \text{for} \quad 0 \leqslant r < g \\ M_r &= B/r - r^2/6 \quad \text{for} \quad g \leqslant r \leqslant v. \end{aligned} \tag{77}$$

Here A and B are constants which are related because of the continuity of M_r

$$B = gA - g^3/48. \tag{78}$$

For the curvatures, we know that $-w'' = k$ in the R_r-region (cf. eqn (62)) and w' for $r \leqslant g$ can be found from eqns (18b). This gives for the curvature κ_θ

$$\begin{aligned} \kappa_\theta &= A - r^2/16 \quad \text{for} \quad 0 \leqslant r < g \\ \kappa_\theta &= k - C/r \quad \text{for} \quad g \leqslant r \leqslant v \end{aligned} \tag{79}$$

where continuity of κ_θ gives that

$$C = kg + g^3/16 - gA. \tag{80}$$

We now employ condition (73) for the optimal placement of the R_r-boundary at $r = g$ and v

$$A - g^2/16 = k - A + 3g^2/16 \tag{81a}$$

$$k - C/v = k - B/v + v^2/6. \tag{81b}$$

From eqn (81a)

$$k = 2A - g^4/4 \tag{82}$$

which inserted in eqn (81b) gives $g^3 = v^3$. We thus see that an R_0-region is only possible at the centre of the plate, and thus $v = t = 1$. The possible optimal simply supported plate must thus be of a type like the intuitive design D. And for this optimal plate, the boundary conditions at $r = 1$ give that $A = (1 + g^3/8)/6g$ and thus (cf. eqns (44))

$$k = (8 - 5g^3)/24g. \tag{83}$$

Finally we note that with these values of A and k the necessary condition $|\kappa_\theta| \leqslant k - |M_r|$ is satisfied in the interior of the R_r-region.

We thus conclude that the optimal simply supported plate is identical with the intuitive design D, with values of Φ_{opt} and g_{opt} shown in Figs 6 and 7.

11. SOLUTION FOR CLAMPED PLATES

In this section, we shall employ the results of the variational analysis in Sections 7–9 to establish optimal solutions for circular plates with clamped edges. We consider types of solutions covered by the following region topography:

(a) *R_0-region* for $r \leqslant g$ ($S_r = S_\theta = 1$),
(b) *R_r-region* for $g < r < t$ ($S_\theta = 0$, $S_r = |M_r|/k$),
(c) *R_0-region* for $t \leqslant r \leqslant 1$ ($S_r = S_\theta = 1$).

As will be shown later, optimal solutions associated with sufficiently small values of $1/C$ correspond to the special case of $t = 1$, that is, they only consist of one (innermost) R_0-region and one R_r-region. For all other relevant values of $1/C$, the topography of the optimal solution will be associated with $t < 1$ and consist of two R_0-regions and one R_r-region. Since the above topography covers both types of relevant designs, it will be used as a basis for our analysis in this sequel.

For the innermost R_0-region, the equilibrium condition (18b) gives

$$\begin{aligned} M_r &= -3r^2/16 + A \\ M_\theta &= -r^2/16 + A \qquad \text{for} \quad 0 \leqslant r < g \\ w' &= r^3/16 - Ar \end{aligned} \tag{84}$$

where A is a constant, see the derivation of eqns (37) and (39).

For the R_r-region, substitution of $S_\theta = 0$ in the equilibrium condition (18b) furnishes $M_r = B/r - r^2/6$, where B is a constant. Denoting the radius of zero radial moment by $r = h$, i.e.

$$M_r(h) = 0 \tag{85}$$

we have $B = h^3/6$, and hence

$$M_r = h^3/6r - r^2/6 \qquad \text{for} \quad g < r < t. \tag{86}$$

Continuity of M_r across $r = g$ then implies $A = (g^3 + 8h^3)/48g$, whereby eqns (84) can be written as

$$\begin{aligned} M_r &= -3r^2/16 + (g^3 + 8h^3)/48g \\ M_\theta &= -r^2/16 + (g^3 + 8h^3)/48g \qquad \text{for} \quad 0 \leqslant r < g. \\ w' &= r^3/16 - r(g^3 + 8h^3)/48g \end{aligned} \tag{87}$$

For the outer R_0-region ($t \leqslant r \leqslant 1$), the equilibrium condition (18b) can be written in the compact form $[(rw')'/r]' = r/2$. Three integrations yield $w' = E/r - Dr/2 + r^3/16$, where E and D are constants. Making use of the boundary condition $w'(1) = 0$, we may express E as $E = D/2 - 1/16$, and we have

$$\left.\begin{aligned} w' &= -D(r^2-1)/2r + (r^4-1)/16r \\ w'' &= -D(r^2+1)/2r^2 + (3r^4+1)/16r^2 \\ M_r &= -w'' = D(r^2+1)/2r^2 - (3r^4+1)/16r^2 \\ M_\theta &= -w'/r = D(r^2-1)/2r^2 - (r^4-1)/16r^2 \end{aligned}\right\} \text{for } t < r \leqslant 1. \tag{88}$$

The condition of continuity of M_r at $r = t$ now gives us the following expression for D upon use of eqn (86) and the third of eqns (88)

$$D = (t^4 + 8th^3 + 3)/[24(t^2+1)]. \tag{89}$$

For the R_r-region ($g < r < t$) with $M_\theta = S_\theta = 0$, eqns (62) prescribe $|\kappa_r| = k$. Since, in general, $M_r = S_r\kappa_r = -S_r w''$, and according to eqn (86) we have $M_r > 0$ for $g \leqslant r < h$ and $M_r < 0$ for $h < r \leqslant t$, it follows that

$$\begin{aligned} \kappa_r &= -w'' = k \quad \text{for} \quad g < r < h \\ \kappa_r &= -w'' = -k \quad \text{for} \quad h < r < t. \end{aligned} \tag{90}$$

We now integrate the second of eqns (90) and impose continuity of w' at $r = t$ by making use of the first of eqns (88). We hereby obtain

$$w'(r) = -D(t^2-1)/2t + (t^4-1)/16t - k(t-r) \qquad \text{for} \quad h < r \leqslant t. \tag{91}$$

Hereafter, we integrate the first of eqns (90) and impose continuity of w' at $r = h$ according to *condition* (76) *for an optimal value of* h. Using eqn (91) for this purpose, we get

$$w'(r) = -D(t^2-1)/2t + (t^4-1)/16t - k(t-2h+r) \qquad \text{for} \quad g \leqslant r \leqslant h. \tag{92}$$

By means of eqn (92), the third of eqns (87) and expression (89) for D, the condition of continuity of w' at $r = g$ leads to the following non-linear algebraic equation between the four unknowns g, h, t and k

$$(4h^3 - g^3)/24 + k(2h - g - t) + (1-t^2)(8h^3t - 2t^4 - 6t^2)/[48t(1-t^2)] = 0. \tag{93}$$

Two additional equations for the unknowns are obtained by using condition (73) for *optimal values of the boundaries* $r = g$ *and* t of R_r-regions. Taking into account the signs of κ_θ and M_r, condition (73) becomes, for $r = g$ and t, respectively

$$\begin{aligned} \kappa_\theta(g) &= k - M_r(g) \\ \kappa_\theta(t) &= k + M_r(t) \end{aligned} \tag{94}$$

where $\kappa_\theta(r) = -w'(r)/r$. Substituting eqns (87)–(89) into (94), we obtain the two equations

$$5g^3 + 24kg - 8h^3 = 0 \tag{95}$$

and

$$3t^5 + 2t^3 + 3t - 8h^3 - 24kt(t^2+1) = 0. \tag{96}$$

It should be noted here that the optimality condition (96) for t presumes that an outer R_r-region is present in the optimal solution, i.e. that $t < 1$. If this is not the case, eqn (96) is to be replaced by the considerably simpler equation $t = 1$.

The *compliance constraint* requires

$$C = \int_0^g (M_r^2 + M_\theta^2) r \, dr + k\left[\int_g^h M_r r \, dr - \int_h^t M_r r \, dr\right] + \int_t^1 (M_r^2 + M_\theta^2) r \, dr. \tag{97}$$

Substituting the relevant expressions for M_r and M_θ into the integrals on the right-hand side and solving for k, we find

$$k = [24C - (5g^6 - 16g^3h^3 + 32h^6)/48 + 6D^2(t^2 - 1/t^2) + 3D(1 - t^4 - t^2 + 1/t^2)/2 - (8 - 5t^6 - 6t^2 + 3/t^2)/32]/[g^4 - 4gh^3 - 4th^3 + 6h^4 + t^4] \tag{98}$$

where D is given by eqn (89).

Equations (93), (95), (96) and (98) determine g, h, t and k for a given value of C. However, if the system of equations does not possess a solution with $t < 1$, eqn (96) is to be replaced by the equation $t = 1$.

It can easily be shown that a design with g, h, t and k determined from the aforementioned equations also satisfies the necessary condition (72), i.e. $|\kappa_\theta| \leqslant k - |M_r|$, in the R_r-region. Hence, the values of g, h, t and k determined from eqns (93), (95), (98) and (96) or $t = 1$ characterize an optimally designed clamped plate.

The *volume* Φ *of the optimal solution* has the form

$$\Phi = \int_0^g r \, dr + \frac{1}{k}\left[\int_g^h M_r r \, dr - \int_h^t M_r r \, dr\right] + \int_t^1 r \, dr \tag{99}$$

and substituting eqn (86) herein, we obtain

$$\Phi = (g^2 + 1 - t^2)/2 + (g^4 - 4gh^3 - 4th^3 + 6h^4 + t^4)/24k. \tag{100}$$

This equation serves to give us Φ when the optimal values of g, h, t and k are determined.

11.1. *Numerical solution procedure and results*

For an appropriately given value of C, eqns (93), (95), (96), (98) and (100) uniquely determine the values of the key parameters g, h, t and k associated with the optimal design possessing the minimum value of Φ. Since the equations are strongly coupled and nonlinear, they require numerical solution. The method developed is based on the following rearrangements of eqns (93) and (95).

Equation (93) can be rewritten in the following form

$$h^3 + 3ph - 2q = 0 \tag{101}$$

giving the real root

$$h = \sqrt[3]{(q + \sqrt{(q + p^3)})} - \sqrt[3]{(-q + \sqrt{(q^2 + p^3)})} \tag{102}$$

where

$$\begin{aligned} p &= 2k(t^2 + 1) \\ q &= (g^3/16 + 3k(g + t)/2)(t^2 + 1) + t(3 - 2t^2 - t^4)/16. \end{aligned} \tag{103}$$

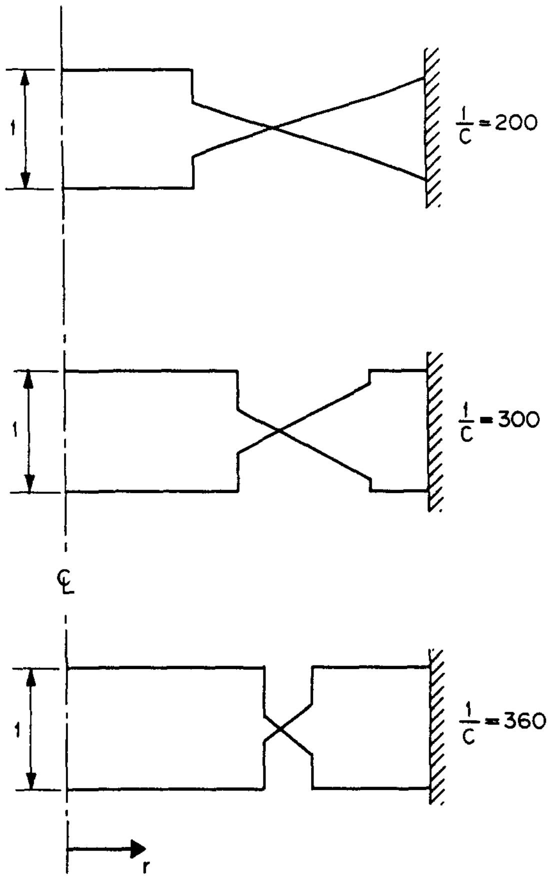

Fig. 8. Optimal rib width distributions for uniformly loaded clamped circular plates.

Similarly, eqn (95) is a cubic equation for g with the real root

$$g = \sqrt[3]{(4/5)}(\sqrt[3]{(h^3 + \sqrt{(h^6 + 32k^3/5)})} - \sqrt[3]{(-h^3 + \sqrt{(h^6 + 32k^3/5)})}). \tag{104}$$

The following iterative scheme was used for the numerical solution procedure:

(0) Assume initial values of g, t, h and k.
(1) Compute h from eqn (102), with p and q given by eqns (103).
(2) Compute k from eqn (98), with D given by eqn (89). If $k < 0$, set $k = 0$. If h and k are not stationary, go to (1).
(3) Solve eqn (96) for t (this is done by a Newton–Raphson procedure). If no solution $0 < t \leqslant 1$ exists, set $t = 1$.
(4) Compute g from eqn (104). If g and t are not stationary, go to (1).
(5) Compute Φ from eqn (100).
End.

The procedure converges rapidly for all prescribed, relevant C-values and uniquely determines the optimal solution both for cases of $t = 1$ and $t < 1$. However, for some C-values it was necessary to introduce an underrelaxation of the new values of t and g computed in steps (3) and (4) for use in the subsequent main iteration loop (1)–(4).

Figure 8 shows examples of distribution of radial stiffness S_r for selected values of $1/C$. The R_0-regions and the R_r-region are easily identified. Note that in the former regions $S_\theta = 1$ and in the latter $S_\theta = 0$.

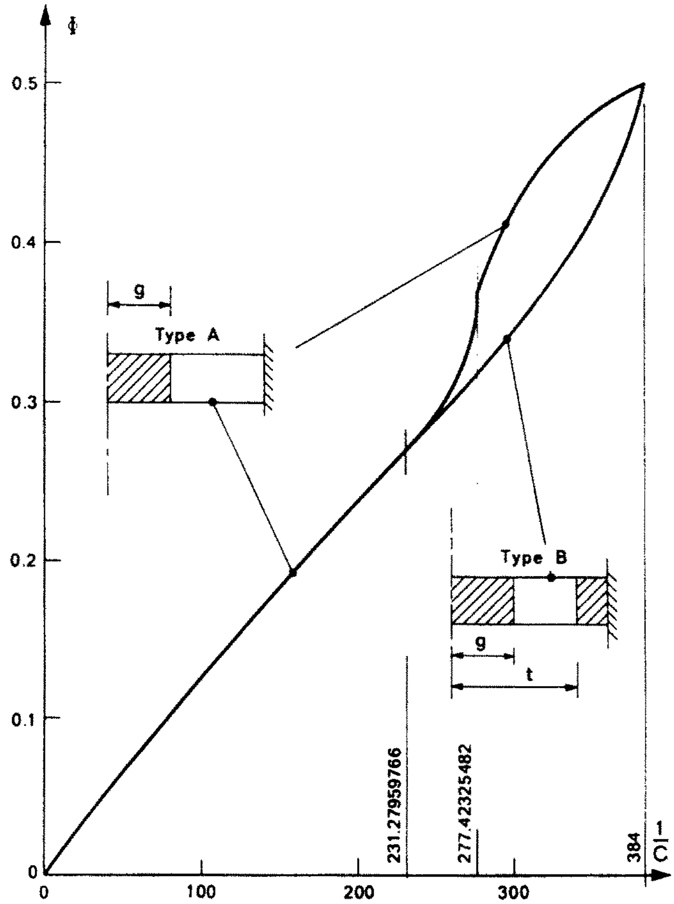

Fig. 9. Optimal weight for "Type A" and "Type B" solutions for uniformly loaded clamped circular plates.

Figure 9 depicts optimum values of Φ as a function of the constraint value $1/C$, and Fig. 10 shows corresponding optimal values of g, h and t.

As indicated in Figs 8 and 9, the optimal clamped plate consists of one (inner) R_0-region and one (outer) R_r-region for sufficiently small values of $1/C$. This type of solution is associated with $t = 1$, and the above conditions (93), (95), (96) and (98) furnish that $t = 1$ requires $1/C < 231.27959766$. For $1/C > 231.27959766$, we have $t < 1$, and the optimal design consists of two R_0-regions and one R_r-region, cf. the two lower parts of Fig. 8.

This state of affairs ceases for $1/C = 384$, where the intermediate R_r-region disappears and the entire plate consists of an R_0-region. This limiting case is easily found to be associated with $g = h = t = 1/\sqrt{3}$ and $\Phi = 1/2$. The case can be checked independently by considering the well-known moment field

$$M_r = (1 - 3r^2)/16, \qquad M_\theta = (1 - r^2)/16 \tag{105}$$

for $S_\theta \equiv S_r \equiv 1$ and $0 \leqslant r \leqslant 1$. Substituting eqns (105) into the compliance constraint then furnishes

$$C = \int_0^1 (M_r^2 + M_\theta^2) r \, \mathrm{d}r = 1/512 + 1/1536 = 1/384 \tag{106}$$

which corresponds to the limiting case just mentioned.

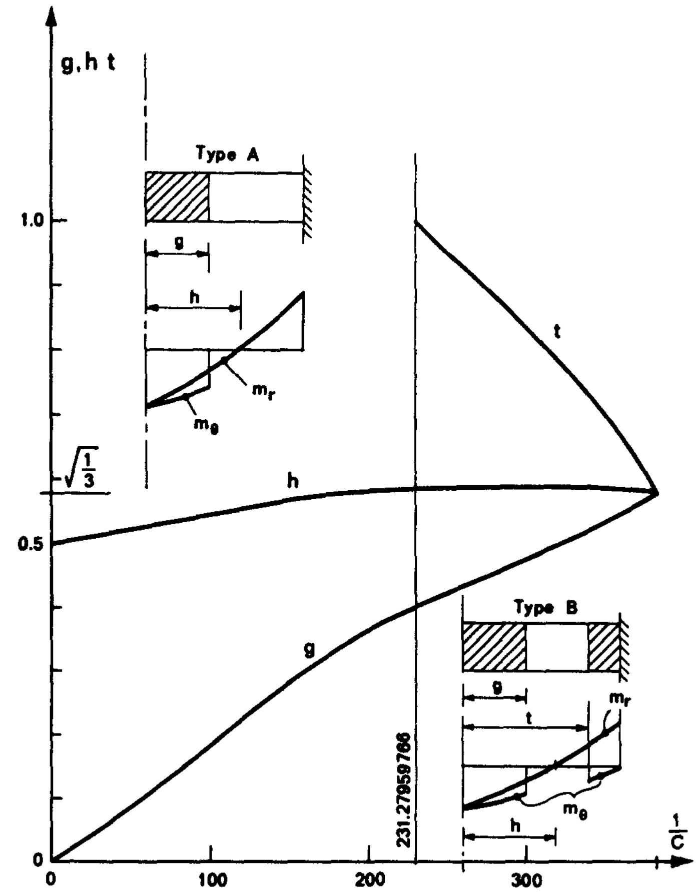

Fig. 10. Optimal values of the region boundary radii (g, t) and the radius of the contraflexure (t) for uniformly loaded clamped circular plates.

12. SOLUTION FOR CIRCULAR PLATES SUBJECTED TO A UNIFORMLY DISTRIBUTED RADIAL COUPLE ALONG THE EDGE

If the circular plate is only loaded by a distributed radial bending moment along the edge, such that we have zero transverse load distribution ($p \equiv 0$), the plate equilibrium equation (see eqn (52)) reduces to

$$(rM_r)'' - M_\theta' = 0 \tag{107}$$

and the conclusions of Sections 8.2 and 8.3 concerning optimal topography are no longer valid. However, the optimality conditions (61)–(63) in connection with the results of Sections 8.1 and 8.4 make it natural to consider the following designs:

Design I : $S_r \equiv S_\theta \equiv S \equiv$ const. and $M_\theta \equiv M_r \equiv 1$ throughout,
Design II : R_0-region for $r \leqslant g$,
R_r-region for $r > g$,
Design III: $S_\theta \equiv S_r \equiv S \equiv$ const. for $r \leqslant g$, R_r-region for $r > g$,

where the boundary radius $r = g$ in Designs II and III is assumed to be optimally placed.

Performing an analysis similar to that of Sections 10 and 11 (but much simpler), we find the interesting result that the above designs all satisfy the necessary conditions (61)–(63) and are of *equal cost* $\Phi = 1/(1 + C)$ for a given value of C.

It is particularly interesting to note that the loading condition considered here leads to optimal plates that contain two-way ribbed regions, cf. Designs I and III with $0 < S < 1$.

However, use of these regions does not result in more efficient plates than application of R_0-regions and R_r-regions.

Finally, we should mention that feasible solutions only exist for $1/C \leqslant 1$, where the limiting case corresponds to Design II with $g = 1$ (i.e. uniform, solid plate).

13. SOLUTION FOR SIMPLY SUPPORTED CIRCULAR PLATES WITH A CENTRAL POINT LOAD

As a final example, we consider the problem of minimizing the weight of a simply supported circular plate loaded by a concentrated force $\bar{P}$ at the centre. It should be noted that this problem is of particular interest because the solution will be optimal for both *prescribed compliance and prescribed maximum deflection.*

Introducing the non-dimensional notation $r = \bar{r}/\bar{R}$, $M_i = \bar{M}_i/\bar{P}\,(i = \theta, r)$, $S_i = 12\bar{S}_i/\bar{E}\bar{h}^3$, $C = \bar{C}\bar{h}^3\bar{E}/24\bar{E}^2\bar{P}^2\pi$, $\Phi = \bar{\Phi}2\bar{h}\bar{R}^2\pi$ (cf. eqns (17)), an integration of the plate equilibrium condition (107) and making use of the load condition at the centre of the plate gives us

$$(rM_r)' - M_\theta = -1/2\pi, \qquad M_r(1) = 0. \tag{108}$$

The optimal plate topography is an R_0-*region* for $0 \leqslant r < g$ and an R_r-*region* for $g < r \leqslant 1$. Equilibrium (108) and optimality conditions furnish the following solution:

$$\left.\begin{aligned} S_r \equiv S_\theta \equiv 1, \qquad M_r &= (1-g)/2g\pi + (1/4\pi)\ln(g/r), \\ M_\theta &= (1-g)/2g\pi + (1/4\pi)[1 + \ln(g/r)], \end{aligned}\right\} \text{for } r < g \tag{109}$$

$$M_\theta \equiv 0, \qquad S_r = |M_r|/k, \qquad M_r = (1-r)/2\pi r, \qquad \text{for } r > g$$

$$\begin{aligned} C &= \int_0^g (M_r^2 + M_\theta^2) r\,dr + \int_g^1 k M_r r\,dr \\ &= (3g^2 - 8g + 8)/32\pi^2 + k(1-g)^2/4\pi, \end{aligned} \tag{110}$$

$$k = 4\pi[C - (3g^2 - 8g + 8)/32\pi^2]/(1-g)^2, \tag{111}$$

$$\Phi = \int_0^g r\,dr + \int_g^1 (|M_r|/k) r\,dr = g^2/2 + (1-g)^2/4\pi k = g^2/2 + (1-g)^4/[4\pi k(1-g)^2] \tag{112}$$

$$|\kappa_\theta(g)| = k - |M_r(g)| \Rightarrow k = (4-3g)/4\pi g. \tag{113}$$

For any C-value, eqns (111) and (113) furnish the optimal values of g and k. The optimal cost Φ is then given by eqn (112)

$$\Phi = g(2 - g^2)/2(4 - 3g). \tag{114}$$

The optimal values of g, k and Φ are given for the feasible range of $(1/C)$ values in Fig. 11.

The limiting value of $(1/C)$ is attained as g takes on its greatest feasible value, i.e. $g = 1$. Relations (110) and (113) then furnish

$$(1/C) = 32\pi^2/3 = 105.2757803. \tag{115}$$

At this compliance value, the plate material fills the entire feasible space. For the same limiting case, relation (113) implies $k = 1/4\pi$.

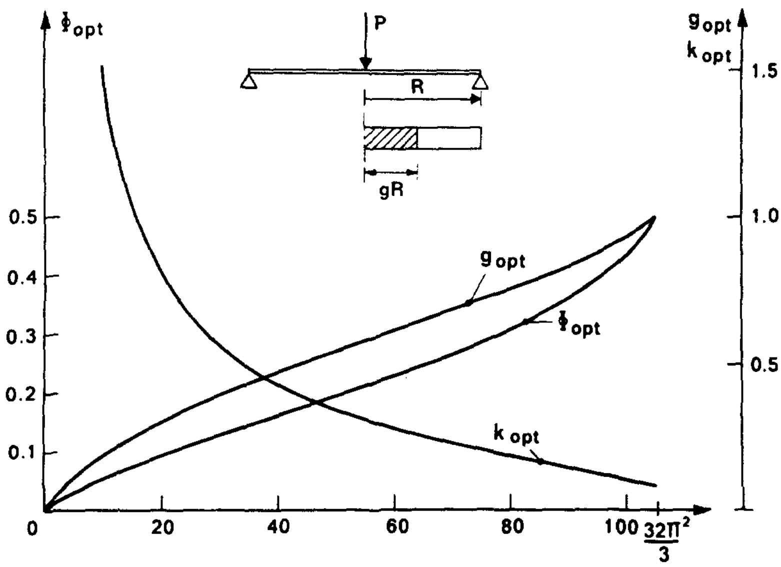

Fig. 11. Optimal values of the total weight Φ, region boundary radius g and the constant k for simply supported plates with a central point load.

14. CONCLUSIONS

(1) Static/kinematic optimality criteria were derived by variational analysis, using the microstructure proposed in Part I of this study. Similar criteria have been proposed by Prager and Shield[23] and used in theories of optimal structural layouts (e.g. Ref. [24]).

(2) The variational formulation has indicated that for transversely loaded axially symmetric plates only two types of optimal regions may occur:

(a) unperforated regions (having a maximum feasible stiffness in both principal directions);

(b) regions consisting of radial ribs only.

For circular plates loaded by a distributed couple along the edge two-way ribbed regions can occur in the optimal plate, but the corresponding minimum cost can also be achieved by a plate made of the kind of regions mentioned above.

We thus conclude that for the class of plates considered, the introduction of the first/second-order microstructure has *not* resulted in an improved economy.

REFERENCES

23. W. Prager and R. T. Shield, A general theory of optimal plastic design. *J. Appl. Mech.* **34**, 184–186 (1967).
24. G. I. N. Rozvany, Structural layout theory—the present state of knowledge, *New Directions in Optimum Structural Design* (Edited by E. Atrek, R. H. Gallagher, K. M. Ragsdell and O. C. Zienkiewicz), Chap. 7. Wiley, Chichester (1984).

APPENDIX: NOTATION

$\bar{C}$	total compliance
$\bar{E}$	Young's modulus
$\bar{h}$	plate depth
$\bar{M}_\theta, \bar{M}_r$	circumferential/radial moments
$\bar{p}$	load intensity
$\bar{r}$	radial coordinate
$\bar{R}$	plate radius
$\bar{S}_\theta, \bar{S}_r$	stiffness in circumferential/radial direction
Φ	total plate volume
$\bar{w}$	plate deflection

Non-dimensional symbols

$C = \bar{C}\bar{h}^3\bar{E}/24\pi\bar{R}^6\bar{p}^2$
$M_i = \bar{M}_i/\bar{p}\bar{R}^2 \quad (i = \theta, r)$
$r = \bar{r}/\bar{R}$
$S_i = 12\bar{S}_i/\bar{E}\bar{h}^3$
$\Phi = \bar{\Phi}\bar{C}\bar{E}\bar{h}^2/24\bar{p}^2\bar{R}^8\pi = \bar{\Phi}/2\bar{h}\bar{R}^2\pi$
$w = \bar{w}\bar{h}^3\bar{E}/12\bar{R}^4\bar{p}$
g radius of region boundary
$\bar{M}$ radial moment at $r = g$
A, B, D, K_1, K_2 constants
$L_1, \ldots, L_4, u$ Lagrangian functions
λ Lagrangian multiplier
$s_1, \ldots, s_4$ slack functions
κ_θ, κ_r circumferential/radial curvature
R_0-regions unperforated regions with $S_r = S_\theta = 1$
R_r-regions regions with radial ribs, $S_\theta \to 0$, $S_r = |M_r|/k$

Specific for *built-in* plates
g, t radii of region boundaries
h radius of points with zero radial moment

Specific for plates with a *central point* load
$\bar{P}$ = point load

The following *non-dimensional* symbols differ from those for plates with distributed loads:

$$M_i = \bar{M}_i/\bar{P}$$
$$C = \bar{C}\bar{h}^3\bar{E}/24\bar{R}^2\bar{P}^2\pi.$$